Stationary Cylindrical Symmetric Solution approaching Einstein's cosmological solution

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Abstract

Here we describe a stationary cylindrically symmetric solution of Einstein's equation with matter consisting of a positive cosmological and rotating dust term. The solution approaches Einstein static universe solution.

1   Introduction

Similar problems for Einstein's equation without cosmological constant and with negative cosmological constant are already solved in literature. In ([1]) a rotating dust cylinder cut out of a Godel universe is matched at exterior to a vacuum stationary cylindrically symmetric solution with negative cosmological constant. In [3] Van Stockum found a rigidly rotating infinitely dust cylinder without cosmological constant which has various exterior metrics.

In this paper we study cylindrically symmetric solutions of the Einstein's field equation with dust and positive cosmological constant which approaches Einstein static universe.

The spatially closed, static Einstein universe in usual form,

$$(1) \qquad \mathrm{d}s_E^2 = \mathrm{d}\eta^2 + \sin^2 \eta (\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\varphi^2) - c^2 \mathrm{d}\psi^2$$
$$\varphi \in [0, 2\pi], \quad \eta \in [0, \pi], \quad \theta \in [0, \pi], \quad \psi \in \mathrm{R}$$

is the simplest cosmological dust model with constant curvature $K = $ const and positive cosmological constant $\Lambda = $ const, $\Lambda > 0$. The field is produced by a energy-momentum tensor T_{ab} of perfect-fluid:

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(2)
$$\kappa T_{ab} = -\Lambda g_{ab} + \mu u_a u_b, \quad \mu > 0, \Lambda = \text{const.} > 0.$$

where $\Lambda = \frac{1}{K^2}$ and $\mu = \frac{2}{K^2} = 2\Lambda = const..$

For the exterior, we will use quite extensively the Einstein metric in cylindrical coordinates:

(3)
$$ds^2 = e^{2V_0(r)}(dr^2 + dz^2) + W_0^2(r)d\varphi^2 - dt^2$$

where $W_0(r)$ and $V_0(r)$ have the form [2]:

(4)
$$V_0(r) = \frac{1}{2}\ln\left(\gamma - \lambda^2\left(\frac{1 - e^{2\lambda(r-\nu)}}{1 + e^{2\lambda(r-\nu)}}\right)^2\right) - \ln\sqrt{\Lambda}$$

(5)
$$W_0(r) = \frac{1 - e^{2\lambda(r-\nu)}}{\left(1 - e^{2\lambda(r-\nu)}\right)^{\frac{\gamma}{\lambda^3}}} e^{2\lambda(r-\nu)\left(\frac{1}{2} - \frac{\gamma}{2\lambda^3}\right)}$$

where γ, α, $\lambda \neq 0$ and ν are constants of integration and $\mu = 2\Lambda = \text{const}$, the dust density respectively.

The space-time of Special Relativity is described mathematicaly by the Minkowski space (M, η). The flat metric η in spherical coordinates [1] has the form:

(6)
$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - c^2 dt^2$$
$$r \in [0, \infty), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi], \quad t \in \text{R},$$

Minkowski spacetime is conformal to a finite region of the Einstein static universe (1).

We have:

(7)
$$ds^2 = \Theta^{-2}ds_E^2,$$

where $\Theta = 2\cos\frac{\psi+\eta}{2}\cos\frac{\psi-\eta}{2}$ is a smooth strictly positive function, under the conformal transformation:

$$t + r = \text{tg}\frac{\psi + \eta}{2}, \quad t - r = \text{tg}\frac{\psi - \eta}{2}, \quad -\frac{\pi}{2} \leq \psi - \eta \leq \psi + \eta \leq \frac{\pi}{2}$$

(the boundaries $\psi \pm \eta = \pm\frac{\pi}{2}$ are the null surfaces $\mathcal{I}^+$ and $\mathcal{I}^-$).

The de Sitter space-times are also conformal to a (finite) part of ds_E^2 and generally, all the closed Robertson-Walker metrics (Minkowski space,

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[1]The coordiantes are singular in $r = 0$ and $\sin\theta = 0$.

the Sitter space are included as special cases) are conformally equivalent to the Einstein static universe. The Robertson-Walker metric,

$$\text{(8)} \qquad \mathrm{d}s^2_{R-W} = K^2(ct)[\mathrm{d}\eta^2 + \sin^2\eta(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)] - c^2\mathrm{d}T^2$$

under the transformation
$$\text{(9)} \qquad \mathrm{d}T = \frac{1}{K(ct)}\mathrm{d}\psi$$

we obtain the Einstein static universe metric (1) .

2 The metric

Stationary gravitational fields are characterized by the existence of a timelike Killing vector field ξ. Therefore in a stationary space-time (M, g) we can construct a global causal structure. In other words we can introduce a coordinate system $(x^a) = (x^\alpha, t)$ with $\xi = \frac{\partial}{\partial t}$. The metric g_{ab} in these coordinates is independent of t and has the general following form:

$$\text{(10)} \qquad \mathrm{d}s^2 = h_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta + F(\mathrm{d}t + A_\alpha\mathrm{d}x^\alpha)^2, \qquad F \equiv \xi_a\xi^a < 0.$$

The unitary timelike vector field $h^0 \equiv (-F)^{-\frac{1}{2}}\xi$ is globally defined on M indicating the time-orientation in every point $p \in M$. Also it gives a global time coordinate t on M.(see [5])

Stationarity (i.e. time translation symmetry) implies that there exists a 1-dimensional group G_1 of isometries ϕ_t whose orbits are timelike curves parametrized by t.

Using the 3-projection formalisme (developed by Geroch (1971)) of a 4-dimensional spacetime manifold (M, g) onto the 3-dimensional differentiable factor manifold $\mathcal{S}_3 = M/G_1$, the Einstein's field equations:

$$\text{(11)} \qquad R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab},$$

for stationary fields take the following simplified form:

$$\text{(12)} \begin{cases} R^{(3)}_{ab} = \dfrac{1}{2}F^{-2}(\dfrac{\partial F}{\partial x^a}\dfrac{\partial F}{\partial x^b} + \omega_a\omega_b) + \kappa(h^c_a h^d_b - F^{-2}\tilde{h}_{ab}\xi^a\xi^b)(T_{cd} - \dfrac{1}{2}Tg_{cd}); \\[2ex] F^{\|a}_{,a} = F^{-1}\tilde{h}_{ab}(\dfrac{\partial F}{\partial x^a}\dfrac{\partial F}{\partial x^b} - \omega_a\omega_b) - 2\kappa F^{-1}\xi^a\xi^b(T_{ab} - \dfrac{1}{2}Tg_{ab}); \\[2ex] \omega^{\|a}_a = 2F^{-1}\tilde{h}_{ab}\dfrac{\partial F}{\partial x^a}\omega_b \\[2ex] F\epsilon^{abc}\omega_{c,b} = 2\kappa h^a_b T^b_c\xi^c \end{cases}$$

Here, "$\|$" denotes the covariant derivative associated with the conformal metric tensor $\tilde{h}_{ab} = -Fh_{ab}$ on $\mathcal{S}_3$ ($h_{ab} = g_{ab} + h_a^0 h_b^0$ is the projection tensor) and $\omega^a = \frac{1}{2}\epsilon^{abcd}\xi_b\xi_{c;d} \neq 0$ [2] is the rotational vector ($\omega^a\xi_a = 0$, $\mathcal{L}_\xi\omega = 0$).

We shall consider that the metric g_{ab} has a cyllindrical symmetry, i.e, it admits as well an Abelian group of isometries G_2 generated by two spacelike Killing vector fields η and ζ, $\mathcal{L}_\eta g_{ab} = \mathcal{L}_\zeta g_{ab} = 0$, $\eta_a\eta^a > 0$, $\zeta_a\zeta^a > 0$ and the integral curves of η are closed (spatial) curves.

There is a theorem (Kundt) which states that an axisymmetric metric can be written in a (2+2)-split if and only if the conditions:

$$(13) \qquad (\eta^{[a}\xi^b\xi^{c;d]})_{;e} = 0 = (\xi^{[a}\eta^b\eta^{c;d]})_{;e}$$

are satisfied.

The existence of the orthogonal 2-surfaces is assured for the dust solutions, provided that the 4-velocity of dust satisfies the condition:

$$(14) \quad u_{[a}\xi_b\eta_{c]} = 0, \quad u^a = (-H)^{-\frac{1}{2}}(\xi^a + \Omega\eta^a) = (-H)^{-\frac{1}{2}}l^i\xi_i^a, \quad \text{where}$$
$$l^i \equiv (1,\Omega), \quad H = \gamma_{ij}l^il^j, \quad \gamma_{ij} \equiv \xi_i^a\xi_{aj}, \quad i,j = 1,2, \quad \xi_1 = \xi; \; \xi_2 = \eta$$

) in other words if the trajectories of the dust lie on the transitivity surfaces of the group generated by the Killing vectors ξ, η. In what follows, we will assume that this is true. Using an adapted coordinate system, the metric (10) can be written in standard form:

$$(15) \qquad \mathrm{d}s^2 = e^{-2U}[e^{2V}(\mathrm{d}r^2 + \mathrm{d}z^2) + W^2\mathrm{d}\varphi^2] - e^{2U}(\mathrm{d}t + A\mathrm{d}\varphi)^2$$

where the functions [3] U, V, W and A depend only on the coordinates (r,z); these coordinates are also conformal flat coordinates on the 2-surface S_2 orthogonal to 2-surface T_2 of the commuting Killing vectors $\xi = \partial_t$ and $\eta = \partial_\varphi$.

If we identify the 4-velocity of the dust u^a with timelike Killing vector $\xi^a = \partial_t = (0,0,0,1)$ then (15) represents a co-moving system ($x^1 = r$, $x^2 = z$, $x^3 = \varphi$, $x^0 = t$) with dust, $u_a = \xi_a = (0, 0, -e^{2U}A, -e^{2U})$ and

$$(16) \qquad \begin{cases} g_{11} = g_{22} = e^{-2U+2V} = h_{11} = h_{22}, \\ g_{33} = e^{-2U}W^2 - e^{2V}A^2 = h_{33}, \quad g_{00} = \xi_0 = -e^{2U} = F, \\ g_{03} = \xi_3 = -e^{2U}A, \quad g_{13} = g_{23} = g_{10} = g_{20} = 0 \end{cases}$$

We can use the complex coordinates $(q,\bar{q})$ on the 2-surface S_2:

$$(17) \qquad\qquad q = \frac{1}{\sqrt{2}}(r + iz)$$

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[2]Here I use the convention: round brackets denote symmetrization and square brackets antisymmetrization and Ω is the angular velocity.

[3]The function W is defined invariantly as $W^2 \equiv -2\xi_{[a}\eta_{b]}\xi^a\eta^b$.

and the stationary axisymmetric metric (15) takes the Lewis- Papapetrou form:

$$(18) \qquad \mathrm{d}s^2 = e^{-2U}(e^{2V}\mathrm{d}q\mathrm{d}\bar{q} + W^2\mathrm{d}\varphi^2) - e^{2U}(\mathrm{d}t + A\mathrm{d}\varphi)^2$$

The surface element on T_2 is $f_{ab} = 2\xi_{[a}\eta_{b]}$, $f_{ab}f^{ab} < 0$ and the surface element on S_2 is $\tilde{f}_{ab}$, the dual tensor of f_{ab}, $\tilde{f}_{ab} = \frac{1}{2}\epsilon_{abcd}f^{cd}$

Thus the Einstein's dust equations with constant $\Lambda > 0$ (12) for the metric (16) will take the following form:

$$\begin{cases} \dfrac{\partial^2 W}{\partial q \partial \bar{q}} = -\Lambda W e^{2V-2U} \\[2ex] \dfrac{\partial^2 U}{\partial q \partial \bar{q}} + \dfrac{1}{2W}\left(\dfrac{\partial U}{\partial q}\dfrac{\partial W}{\partial \bar{q}} + \dfrac{\partial U}{\partial \bar{q}}\dfrac{\partial W}{\partial q}\right) + \dfrac{1}{2W^2}e^{4U}\dfrac{\partial A}{\partial q}\dfrac{\partial A}{\partial \bar{q}} = (\mu - 2\Lambda)\dfrac{e^{2V-2U}}{4} \\[2ex] \dfrac{\partial^2 A}{\partial q \partial \bar{q}} - \dfrac{1}{2W}\left(\dfrac{\partial A}{\partial q}\dfrac{\partial W}{\partial \bar{q}} + \dfrac{\partial A}{\partial \bar{q}}\dfrac{\partial W}{\partial q}\right) + 2\left(\dfrac{\partial A}{\partial q}\dfrac{\partial U}{\partial \bar{q}} + \dfrac{\partial A}{\partial \bar{q}}\dfrac{\partial U}{\partial q}\right) = 0 \\[2ex] \dfrac{\partial^2 W}{\partial q \partial \bar{q}} - 2\dfrac{\partial W}{\partial q}\dfrac{\partial V}{\partial q} + 2W\left(\dfrac{\partial U}{\partial q}\right)^2 - \dfrac{1}{2W}e^{4U}\left(\dfrac{\partial A}{\partial q}\right)^2 = 0 \\[2ex] \dfrac{\partial^2 V}{\partial q \partial \bar{q}} + \dfrac{\partial U}{\partial q}\dfrac{\partial U}{\partial \bar{q}} + \dfrac{1}{(2W)^2}e^{4U}\dfrac{\partial A}{\partial q}\dfrac{\partial A}{\partial \bar{q}} = -\Lambda\dfrac{e^{2V-2U}}{2} \end{cases}$$

(19)

Here $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{\partial^2}{\partial z^2} = 2\dfrac{\partial^2}{\partial q \partial \bar{q}}$ is the Laplace operator and the energy-momentum tensor T_{ab} has the form (2) with $\Lambda = const. > 0$ and $\mu(r) > 0$.

The conservation law $T^{ab}_{;b} = 0$ implies $U_{,a} = 0$. We obtain then $U = const.$ a consequence of the field equations which will use it in place of one of the Einstein's equations.

Moreover assuming that $U = 0$ in the expressions of metric functions (16) and we obtain that the matter current paths are geodesics ($\dot{u}_a = u_{a;b}u^b = 0$), without expansion ($\theta = u^a_{;a} = 0$), in a non-rigidly rotation ($\omega = \sqrt{\frac{1}{2}\omega_{ab}\omega^{ab}} \neq 0$) and with $\sigma \neq 0$.

The field equations (19) will take then the following simplified form:

$$(20) \quad \begin{cases} \dfrac{\partial^2 W}{\partial q \partial \bar{q}} = -\Lambda W e^{2V} \\[2mm] \dfrac{1}{2W^2} \dfrac{\partial A}{\partial q} \dfrac{\partial A}{\partial \bar{q}} = (\mu - 2\Lambda) \dfrac{e^{2V}}{4} \\[2mm] \dfrac{\partial^2 A}{\partial q \partial \bar{q}} - \dfrac{1}{2W} \left(\dfrac{\partial A}{\partial q} \dfrac{\partial W}{\partial \bar{q}} + \dfrac{\partial A}{\partial \bar{q}} \dfrac{\partial W}{\partial q} \right) = 0 \\[2mm] \dfrac{\partial^2 W}{\partial q \partial \bar{q}} - 2 \dfrac{\partial W}{\partial q} \dfrac{\partial V}{\partial q} - \dfrac{1}{2W} \left(\dfrac{\partial A}{\partial q} \right)^2 = 0 \\[2mm] \dfrac{\partial^2 V}{\partial q \partial \bar{q}} + \dfrac{1}{(2W)^2} \dfrac{\partial A}{\partial q} \dfrac{\partial A}{\partial \bar{q}} = -\Lambda \dfrac{e^{2V}}{2} \end{cases}$$

Taking into account the third symmetry i.e., the presence of the spacelike Killing vector field $\zeta = \partial_z$ we reduced to the problem of solving the Einstein's system of ordinary differential equations (20) for the metric:

$$\mathrm{d}s^2 = e^{2V(r)}(\mathrm{d}r^2 + \mathrm{d}z^2) + W^2(r)\mathrm{d}\varphi - (\mathrm{d}t + A(r)\mathrm{d}\varphi)^2$$

in the unknown metric functions $V(r)$, $W(r)$, $A(r)$ and $\mu(r)$ and to match the constant of integration such that the exterior field is conformal with Einstein static universe [4]

If we denote $\frac{\partial}{\partial r} ='$ the field equations (20) will take the simplified form:

$$(21) \quad \begin{cases} W'' = -2\Lambda W e^{2V} \\[1mm] 2A'^2 = (\mu - 2\Lambda)W^2 e^{2V} \\[1mm] \dfrac{A''}{A'} = 2\dfrac{W'}{W} \\[1mm] W'' - 4W'V' - \dfrac{1}{W}A'^2 = 0 \\[1mm] V'' + \dfrac{1}{2W^2}A'^2 = -\Lambda e^{2V} \end{cases}$$

The system of equations (21) can be further reducible to the following form:

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[4]For the case when $\Lambda = 0$ has an exterior static ($\omega^a = \frac{1}{2}\epsilon^{abcd}\xi_b\xi_{c;d} = 0$) even the dust was in rotation with $\Omega = $ const. (Van Stockum class solutions).

(22)
$$\begin{cases} W'' = -2\Lambda W e^{2V} \\ A' = aW^2 \\ (\mu - 2\Lambda)e^{2V} = 2a^2 W^2 \\ W'' - 4W'V' - a^2 W^3 = 0 \\ W'' - a^2 W^3 - 2V''W = 0 \end{cases}$$

where $a \neq 0$, $b \neq 0$ are positive constants and Λ is the positive cosmological constant, which is very small (less than $10^{-57}cm^{-2}$).

After further simplifications the system of equations (22) will take the form:

(23)
$$\begin{cases} \dfrac{W''}{W} = -2\Lambda e^{2V} \\ A' = aW^2 \\ V' = bW^2 \\ (\mu - 2\Lambda)e^{2V} = 2a^2 W^2 \\ W'' - 4bW'W^2 - a^2 W^3 = 0 \end{cases}$$

where a and b are positive constants of integration.

The system (23) does not have an explicit analytical solution for $W(r)$, $V(r)$, $A(r)$ and $\mu(r)$ as functions of radius r. Therefore we shall look to derive a good approximation of the solution.

We remark from the form of the system, that we are looking for a one-parameter[5] family $g_{ij}(a)$ of solutions, where a measures the size of perturbation, in the sense that $g_{ij}(a)$ are continuous differentiable on a and for $a = 0$ we obtain Einstein universe solution.

In what follows we shall show that the the solution of (23) is approaching Einstein universe solution $g_{ij}(r, 0)$, as radius r goes to zero.

(24) $\qquad g_{ij}(r, a) = g_{ij}(r, 0) + a g_{ij,a}(r, 0) + a^2 g_{ij,aa}(r, 0) + ...$

We shall perturb the solution as power series in a about Einstein universe solution and give a good approximation to $g_{ij}(a)$ for sufficiently small a.

To do so we differentiate the system (23) with respect to a, then take a to be zero and obtain the following equations:

(25)
$$\begin{cases} \dot{W}''(r, 0) - P(r)\dot{W}'(r, 0) - Q(r)\dot{W}(r, 0) = 0 \\ \dot{A}'(r, 0) = W_0^2 \\ \dot{\mu}(r, 0)e^{2V_0} = 0 \\ \dot{V}'(r, 0) = 2bW_0\dot{W}(r, 0) \end{cases}$$

[5]It is actually a two-parameter family of solutions $g_{ij}(a, b)$

for the functions $\dot{W}(r,0)$, $\dot{V}(r,0)$, $\dot{\mu}(r,0)$ $\dot{A}(r,0)$.

We denoted by $\frac{\partial}{\partial a} = \dot{}$, $P(r) = 4bW_0^2$, $Q(r) = 8bW_0W_0'$ and $W_0(r)$, $V_0(r)$ are the metric functions of Einstein universe solution (3).

By choosing appropriate constants of integration $\nu = 0$, $\lambda = 1$, $\gamma = 1$ in (5) we get $W_0(r) = 1$. [2]

Then the system (25) can be completely integrated and take the following form:

(26)
$$\begin{cases} \dot{W}(r,0) = c_1 + c_2 e^{4br} \\ \dot{A}(r,0) = r(1 + 2ac_1) + \dfrac{ac_2}{2b} e^{4br} \\ \dot{\mu}(r,a) = 0 \\ \dot{V}(r,0) = 2bc_1 r + \dfrac{c_2}{2} e^{4br} \end{cases}$$

Then $g_{ij}(r,0) + ag_{ij,a}(r,0)$ will give a good approximation to solution of the Einstein field equations (23) $g_{ij}(r,a)$ for small a when r approaches the axis of rotation $\eta = 0$.

(27)
$$\begin{cases} W(r,a) = 1 + ac_1 + ac_2 e^{4br} \\ A(r,a) = ar(1 + 2ac_1) + \dfrac{a^2 c_2}{2b} e^{4br} \\ \mu(r,a) = 2\Lambda \\ V(r,a) = V_0(r) + 2abc_1 r + \dfrac{ac_2}{2} e^{4br} \end{cases}$$

3 Conclusion

In this paper we have been studying a spacetime satisfying Einstein field equations with positive cosmological constant, describing a dust cylinder in non-rigid rotation, which approaches Einstein's cosmological static solution on the axis of rotation. The metric is given in approximation around the axis of rotation and it depends on three parameters a, b and Λ.

4 Acknowledgements

References

[1] **BONNOR, W B, SANTOS, N O, MACCALLUM MAH**, *An exterior for the Godel spacetime*, Class. Quantum Grav. 15, (1998)

[2] **IFTIME, M.D.**, *Static Cylindrically Symmetric Solution*,Annals of Physics, Timisoara, Ro(1999),

[3] **IFTIME, M.D.**, *Spacetime distribution*, Southwest Journal of Pure and Applied Mathematics, Oklahoma, USA, (1999),

[4] **VAN STOCKUM**, , Proc. R. Soc Edin, 57 (1937)

[5] **KRAMER, D., STEPHANI, H., HERLT, E., MACCALLUM, M., SCHMUTZER, E.**, *Exact solutions of Einstein's field equations*, Cambridge Univ. Press, (1980)